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# DYNAMICAL PROBLEMS OF CONTINUOUS MEDIA WITH RANDOM BOUNDARY DATA†

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**Abstract**—A spatial correlation method is formulated for linear dynamical problems in continuum mechanics with random boundary data. The essential feature of the method is the formulation of a nonstochastic mixed initial-boundary value problem for the (matrix) spatial correlation function of the (vector) state variable. Whenever the Green's function of the (stochastic) problem can not be obtained in terms of known functions, a numerical solution of the meansquare response and other second order response statistics by the spatial correlation method is several hundred folds more efficient than any other available method. Further improvements in the computational efficiency of the method for a steady state stationary response process are also noted.

## 1. INTRODUCTION

Linear dynamical problems of continuous media are usually formulated as initial-boundary value problems (IBVP) for linear partial differential equations. We are interested here in the solution of such problems when the distributed loading as well as the prescribed initial and boundary data are random functions of position and time with known statistics. A solution in that case consists of obtaining the joint moment functions (or joint probability density functions) of all orders of the state variable(s). In practice, one very often settles for the relevant first and second order statistics.

Conventional methods (based on the $L_2$ calculus) for computing these statistics may be divided into two distinct groups. The first group seeks the formal solution of the stochastic IBVP in terms of the random loadings and auxiliary data (e.g. the Green's function representation in the time or frequency domain), and then use it to get, by appropriate ensemble averaging, the response statistics in terms of the known load statistics. A discussion of this group of methods, collectively called the *Green's function method* here, can be found in [1, 2] and references therein. The essential feature of the second group of methods, collectively called the *direct method* here, is the formulation of nonstochastic IBVP for the desired statistics themselves. These nonstochastic IBVP are then solved by conventional methods of applied mathematics. Discussion of this second approach may be found in [1, 3, 4] and references therein. Whenever the Green's function of the original (stochastic) IBVP can be found in terms of known functions, there is no essential difference between these two approaches in the level of computation involved.

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In this paper, we are concerned with problems for which the *relevant Green's function can not be found in terms of elementary or special functions.* One example of such problems is the forced transverse random vibration of flexible lifting rotor blades in forward flight. The transverse displacement of the blade normalized by the blade length is governed by the dimensionless PDE

$$\zeta^4 w_{xxxx} - \tfrac{1}{2}(1 - x^2)w_{xx} + [x + \gamma_0 \mu \cos t|x + \mu \sin t|]w_x + [\beta + \gamma_0|x + \mu \sin t|]w_t + w_{tt} = g(x, t) \qquad (0 < x < 1, t > 0) \quad (1.1)$$

where $t$ and $x$ are the dimensionless time and distance from the blade root, $\zeta$, $\gamma_0$, $\mu$ and $\beta$ are known constants and where $g(x, t)$ is a random process with known statistics. The statistics of complex random response processes such as the blade response $w(x, t)$, can only be obtained by some numerical method. For such complex processes, the direct approach seems preferable since it is difficult to obtain accurate numerical solutions of Green's functions in the physical space, not to mention the amount of machine computation involved. However, except for special cases, the available direct methods for second and higher order statistics also require an unrealistic amount of machine calculation and are thus also impractical.

To bring out the computational efficiency (in connection with a numerical solution) inherent in the direct approach not realized previously, a new method, effective and practical for a much wider class of problems which includes the rotor blade problem, was developed recently in[4, 5]. For simplicity, this "spatial correction method" was formulated there for problems with homogeneous initial and boundary conditions. In principle, problems with inhomogeneous auxiliary conditions can be transformed into one with homogeneous auxiliary conditions. For example, the problem

$$u_t = u_{xx} \qquad (0 < x < 1, t > 0) \quad (1.2)$$

with

$$u(x, 0) = 0 \qquad (0 \le x \le 1) \quad (1.3)$$

$$u(0, t) = 0, \qquad u_x(1, t) = f(t) \qquad (t > 0) \quad (1.4)$$

can be transformed into a problem with vanishing auxiliary conditions by setting $v(x, t) = u(x, t) - xf(t)$. In terms of $v(x, t)$, we have

$$v_t = v_{xx} - xf_t \qquad (0 < x < 1, t > 0) \quad (1.5)$$

with (taking $f(0) = 0$)

$$v(x, 0) = v(0, t) = v_x(1, t) = 0. \quad (1.6)$$

Even in cases where $f(t)$ is a known function, the conversion is possible only if $f$ is differentiable. For a random function $f(t)$ with only a few sample histories, it seems even more desirable to work directly with the original problem (1.2)–(1.4).

In this paper, a spatial correlation method will be formulated for problems with random boundary data. More specifically, we will formulate a numerically efficient solution scheme for the second order statistics of the solution of the IBVP

$$u_t = L[u] \qquad (x \in V, t > 0) \quad (1.7)$$

with

$$u(x, 0) = 0 \qquad (x \in \bar{V}) \quad (1.8)$$

$$B[u] = f(x, t) \qquad (x \in S, t > 0). \quad (1.9)$$

Here, $u$ is a vector function defined in some volume $V$ bounded by some surface(s) $S$† (with $\bar{V}$ being the union of $V$ and $S$) and all time $t \geq 0$. $L$ and $B$ are matrix linear partial differential operators involving only spatial derivatives, though the known coefficients of these operators may be continuous functions of both $x$ and $t$. For example, by setting $u_1 = w$ and $u_2 = w_t$, equation (1.1) can be written in the form (1.7) with coefficients of $L$ being continuous functions of both $x$ and $t$.

Upon ensemble averaging both sides of (1.7)–(1.9), we find that the expected value of the response $u$ is determined by the original IBVP with $f(x, t)$ in (1.9) replaced by its expected value. In view of the linearity of the problem, we will henceforth take $f(x, t)$ to be of zero mean so that $u$ is also of zero mean. We can therefore concentrate on the second order statistics of $u$ characterized by its autocorrelation matrix.

While the method to be developed herein is very much akin to that described in[4, 5] for random forcing distributed throughout $V$ (henceforth called distributed loading), there are enough distinct features in the analysis and results to deserve a separate treatment, at least up to the point when further development becomes almost a repetition of [5].‡ Just as in [5], our method is mainly for problems which can not be solved by known functions, i.e. the Green's function of (1.7)–(1.9) can not be found in terms of elementary or special functions. To obtain the steady state meansquare response by numerical integration via our spatial correlation method is known to be several hundred folds more efficient than any other available method[6, 7].

## 2. TEMPORALLY UNCORRELATED RANDOM BOUNDARY VALUES

Guided by the results of[4, 5], we consider in this section the class of linear dynamical problems, (1.7)–(1.9), with temporally uncorrelated random boundary data. More specifically, the vector random function $f(x, t)$ in (1.9) is to be of zero mean and with an autocorrelation function

$$\langle f(x, t) f^T(y, \tau)\rangle = R_S(x, y, \tau)\delta(t - \tau) \qquad (x, y \in S) \tag{2.1}$$

where $(\ )^T$ is the transpose of $(\ )$, $\delta(t)$ is the Dirac delta function, and $R_S{}^T(y, x, t) = R_S(x, y, t)$. We begin by formulating an IBVP (initial-boundary value problem) for the spatial correlation matrix $U(x, y, t)$ of the unknown vector random function $u(x, t)$,

$$U(x, y, t) = \langle u(x, t) u^T(y, t)\rangle. \tag{2.2}$$

Evidently, $U$ is defined in the product space $\bar{V} \times \bar{V} \times (0, \infty)$. The solution of this problem will then be used as the initial condition for another IBVP which determines the autocorrelation matrix $R(x, t; y, \tau) = \langle u(x, t) u^T(y, \tau)\rangle$. The latter completely characterizes the second order statistics of $u(x, t)$.

### 2.1 *The spatial correlation matrix*

To get a partial differential equation for the spatial correlation matrix $U$, we ensemble average the identity $[u(x, t)u^T(y, t)]_t = u_t(x, t)u^T(y, t) + u(x, t)u_t{}^T(y, t)$ after we use equation (1.7) to eliminate $u_t$ and $u_t{}^T$. The result is

† For a multiply connected $V$, equation (1.9) denotes the collection $\{B_i[u] = f_i(x, t), (x \in S_i, t > 0)\}$.

‡ In contrast, the formulation of a spatial correlation method for a problem with no distributed loading and homogeneous boundary conditions but with random initial data of known statistics involves no additional novel feature and therefore will not be reported.

$$U_t = L_x[U] + (L_y[U^T])^T \tag{2.3}$$

where the subscripts $x$ and $y$ indicate that we are working in the $x$, $t$– and $y$, $t$–space, respectively. This equation is supplemented by the initial condition

$$U(x, y, 0) = 0, \tag{2.4}$$

which follows from (1.8), and by the boundary conditions

$$B_x[U] = F(x, y, t) \qquad (x, y) \in (S \times \bar{V}) \tag{2.5}$$

$$B_y[U^T] = F(y, x, t) \qquad (x, y) \in (\bar{V} \times S), \tag{2.6}$$

which follow from (1.9) with $F(x, y, t) = \langle f(x, t)u^T(y, t)\rangle$.

For a temporally uncorrelated $f(x, t)$, we get from (2.1) and the Green's function representation of the solution of (1.7)–(1.9),

$$u(y, t) = \int_S \int_0^t G(y, t \mid y', t') f(y', t')\, \mathrm{d}t'\, \mathrm{d}y' \tag{2.7}$$

the relation

$$\begin{aligned} F(x, y, t) &= \int_S \int_0^t R_S(x, y', t')\, \delta(t - t') G^T(y, t \mid y', t')\, \mathrm{d}t'\, \mathrm{d}y' \\ &= \tfrac{1}{2} \int_S R_S(x, y', t) G^T(y, t \mid y', t)\, \mathrm{d}y'. \end{aligned} \tag{2.8}$$

Since $G(y, t \mid y', t) = I\delta(y - y')$ (see Appendix of [5]), we have

$$F(x, y, t) = \begin{cases} 0 & y \notin S \\ \frac{1}{2} R_S(x, y, t) & y \in S \end{cases}. \tag{2.9}$$

With (2.9), the IBVP (2.3)–(2.6) completely determines $U(x, y, t)$ and can be solved, exactly or approximately, by known methods of applied mathematics. In particular, a numerical solution can always be obtained in a straightforward manner whenever the Green's function of the problem (1.7)–(1.9) can not be obtained in terms of known function. A numerical solution of the meansquare properties of the response, contained in the covariance matrix $U(x, x, t)$, by way of (2.3)–(2.6) is several hundred folds more efficient than any other available method (see[7] for a more detailed discussion on this point). This computational efficiency constitutes the *raison d'être* for the development of the spatial correlation method.

Finally, we note that the inhomogeneous term $F(x, y, t)$ for temporally uncorrelated boundary data is always a discontinuous function even when $R_S(x, y, t)$ is continuous. This is in contrast to the case of uncorrelated distributed loading where the IBVP for $U(x, y, t)$ involves only continuous inhomogeneous terms if $R_S(x, y, t)$ is continuous.

### 2.2 *The autocorrelation matrix*

To formulate an IBVP for the autocorrelation matrix $R(x, t; y, \tau)$, we post-multiply (1.7)–(1.9) by $u^T(y, \tau)$ and ensemble average to get

$$R_t - L_{xt}[R] = 0 \tag{2.10}$$

$$R(x, 0; y, \tau) = 0 \tag{2.11}$$

$$B_{xt}[R] = \langle f(x, t)u^T(y, \tau)\rangle \equiv C(x, t; y, \tau) \tag{2.12}$$

where the subscripts of $L$ and $B$ are to indicate that we are working in the $x$, $t$–space with $y$ and $\tau$ as parameters. We can of course formulate another IBVP for the unknown cross-correlation matrix $C(x, t; y, \tau)$ by post-multiplying (1.7)-(1.9) by $f^T(y, \tau)$ and ensemble averaging the resulting equations. Upon interchanging the roles of $(x, t)$ and $(y, \tau)$, we get

$$C_\tau^T - L_{y\tau}[C^T] = 0 \tag{2.13}$$

$$C(x, t; y, 0) = 0 \tag{2.14}$$

$$B_{y\tau}[C^T] = \langle f(y, \tau) f^T(x, t) \rangle. \tag{2.15}$$

Since the right side of (2.15) is known, we may first solve (2.13)–(2.15) for $C(x, t; y, \tau)$ in the $y$, $\tau$–space with $x$ and $t$ as parameters and then use the result in (2.12) for the determination of $R$. While this procedure (equivalent to the *autocorrelation method* for distributed loading described in[3, 4, 5]) is straightforward in principle, it is impractical whenever a solution by numerical integration is necessary. For a fixed pair of $(y_0, \tau_0)$, we must solve the problem (2.13)–(2.15) once numerically (for all $\tau \leq \tau_0$) for every $t_n$ used in a step-by-step integration scheme for the determination of $R$.

To formulate a more efficient procedure for the numerical solution of $R$, we note that, for a temporally uncorrelated $f(x, t)$, we have from the Green's function representation (2.7)

$$\begin{aligned} C(x, t; y, \tau) &= \int_S \int_0^t \langle f(x, t) f^T(y', \tau') \rangle G^T(y, \tau \mid y', \tau')\, d\tau'\, dy' \\ &= H(\tau - t) \int_S R_S(x, y', t) G^T(y, \tau \mid y', t)\, dy' \end{aligned} \tag{2.16}$$

where $H(t)$ is the unit step function. So, if (2.10) is solved for $t > \tau$ only, i.e.

$$R_t - L_{xt}[R] = 0 \qquad (t > \tau), \tag{2.17}$$

we have, instead of (2.12), the homogeneous boundary condition

$$B_{xt}[R] = 0 \qquad (x, y) \in (S \times \bar{V}), (t > \tau). \tag{2.18}$$

But now as the initial condition, we have, instead of (2.11), the inhomogeneous condition

$$R(x, \tau; y, \tau) = U(x, y, \tau) \tag{2.19}$$

which follows from the definition of the two quantities involved and the continuity of $R$ across $t = \tau$. But $U(x, y, \tau)$ has already been obtained in section (2.1); equations (2.17)–(2.19) therefore define an IBVP for $R$. Once we have $R(x, t; y, \tau)$ for $t > \tau$, we can obtain $R$ for $t < \tau$ from the symmetry condition

$$R(x, t; y, \tau) = R^T(y, \tau; x, t). \tag{2.20}$$

### 2.3 *Heat flow in a rod with temporally uncorrelated end flow rate*

In the development of section (2.1), we have tacitly avoided problems which are one dimensional in the spatial variable. While our method goes through even for this omitted case, a special feature of the IBVP for the spatial correlation $U(x, y, t)$ should be pointed out. We will do this here by way of the scalar heat flow problem, (1.2)–(1.4), for a zero mean, temporally uncorrelated $f(t)$ with

$$\langle f(t) f(\tau) \rangle = q(\tau) \delta(t - \tau). \tag{2.21}$$

Upon specializing (2.3)–(2.6) (which hold also for one dimensional problems) to the heat flow problem, we have

$$U_t = U_{xx} + U_{yy} \qquad (0 < x, y < 1, t > 0) \tag{2.22}$$

$$U(x, y, 0) = 0 \qquad (0 \le x, y \le 1) \tag{2.23}$$

$$U(0, y, t) = 0 \qquad U_x(1, y, t) = F(y, t) \qquad (0 \le y \le 1, t > 0) \tag{2.24}$$

$$U(x, 0, t) = 0, \qquad U_y(x, 1, t) = F(x, t) \qquad (0 \le x \le 1, t > 0) \tag{2.25}$$

where $F(z, t) = \langle f(t)u(z, t)\rangle$. From the Green's function representation of the solution of (1.2)–(1.4),

$$u(x, t) = \int_0^t G(x, t \mid 1, t')f(t')\,\mathrm{d}t' \tag{2.26}$$

we get

$$\begin{aligned} F(z, t) &= \int_0^t G(z, t \mid 1, t')\langle f(t')f(t)\rangle\,\mathrm{d}t' \\ &= \tfrac{1}{2}G(z, t \mid 1, t)q(t) = \tfrac{1}{2}\delta(z - 1)q(t). \end{aligned} \tag{2.27}$$

With (2.27), equations (2.22)–(2.25) define an IBVP which is the same as the problem of temperature distribution in a unit square plate with a hot spot at one of its corners. While the solution of this problem is straightforward, the inhomogeneous terms in the IBVP for $U$ are not ordinary functions, in contrast to the higher dimensional case discussed in section (2.1).

For the special case $q(\tau) = q_0$ (a constant), the steady state response of (1.2)–(1.4) is known to be a stationary process. Within the framework of our spatial correlation method, this is reflected in the fact that the steady state solution $\hat{U}$ of (2.22)–(2.25) is independent of $t$ and may be obtained by solving the boundary value problem (BVP)

$$\hat{U}_{xx} + \hat{U}_{yy} = 0 \qquad (0 < x, y < 1) \tag{2.28}$$

with

$$\begin{aligned} &\hat{U}(0, y) = 0, \qquad \hat{U}_x(1, y) = \tfrac{1}{2}q_0\,\delta(y - 1). \\ &\hat{U}(x, 0) = 0, \qquad \hat{U}_y(x, 1) = \tfrac{1}{2}q_0\,\delta(x - 1). \end{aligned} \tag{2.29}$$

A solution of (2.28)–(2.29) by separation of variables is immediate. We omit the routine calculations and give here only the final result

$$\hat{U}(x, y) = \sum_{n=0}^{\infty} \frac{q_0(-1)^n}{\lambda_n \cosh \lambda_n}\,[\sinh(\lambda_n x)\sin(\lambda_n y) + \sinh(\lambda_n y)\sin(\lambda_n x)] \tag{2.30}$$

where $\lambda_n = \frac{1}{2}(2n + 1)\pi$. The steady state meansquare temperature is then obtained by setting $y = x$ in (2.30). The single series (2.30) for $\hat{U}(x, y)$ by our spatial correlation method seems less cumbersome than the double Fourier series solution obtained by a more conventional Green's function method with the Green's function expressed in terms of the relevant normal modes[1].

For a more complex dynamical process which is stationary in its steady state and for which a numerical solution of the steady state spatial correlation is necessary, it is much simpler (and less expensive) to obtain this steady state solution by way of a BVP instead of the corresponding IBVP (see[6] for a discussion on this point).

## 3. SHAPED FILTERED SHOT NOISE DATA

As a step toward removing the restriction of temporally uncorrelated data, we consider in this section the class of boundary data with a random part independent of $x$. That is, we have $f(x, t) = E(x, t)n(t)$ where $E(x, t)$ is a known envelope (matrix) function and $n(t)$ is a zero mean random function. Furthermore, we assume that $n(t)$ has the same first and second order statistics as the steady state response of some linear dynamical system characterized by the vector equation

$$n_t = A(t)n + D(t)w(t) \tag{3.1}$$

where $A$ and $D$ are known matrix functions of $t$ and $w(t)$ is a zero mean vector random process with

$$\langle w(t)w^T(\tau)\rangle = Q(\tau)\delta(t-\tau), \tag{3.2}$$

$Q$ being a symmetric positive semi-definite matrix. The relation (3.2) implicitly assumes that a component of $n(t)$ is the response to shot noise of a filter of order not higher than the dimension of $n(t)$ itself. If this is not true, then $w(t)$, instead of being uncorrelated, should itself be taken as the response to shot noise of yet another linear dynamical system of order not higher than the dimension of $w$, and so on. We will consider in what follows only the case where a single equation of the form (3.1) suffices since no new element is introduced to our method of solution by additional equations of the form (3.1).

Since $n(t)$ is not itself delta correlated, an important relation analogous to (2.9) is no longer available. We must now devise a new method to obtain the unknown $F(x, y, t)$ in (2.5) and (2.6) to complete the formulation of the IBVP for the determination of the spatial correlation matrix.

Similarly, the cross-correlation $C(t;\, y, \tau) = \langle n(t)u^T(y, \tau)\rangle$ does not vanish for $t > \tau$ in the case of a correlated $n(t)$. The procedure of section (2.2) for the determination of the auto-correlation matrix must also be modified.

### 3.1 *The spatial correlation matrix*

The solution of (3.1) can be expressed in terms of the associated fundamental (or impulse response) matrix $h(t, z)$:

$$n(t) = \int_{-\infty}^{t} h(t, z)D(z)w(z)\,\mathrm{d}z. \tag{3.3}$$

From (3.3) follows the relation

$$\langle n(t)w^T(t')\rangle = H(t-t')h(t, t')D(t')Q(t') \tag{3.4}$$

which in turn implies (with the help of (2.7))

$$\langle w(t)u^T(x, t)\rangle = \langle u(x, t)w^T(t)\rangle = 0. \tag{3.5}$$

Now we form the ensemble average of the identity $[u(x, t)n^T(t)]_t = u_t n^T + u n_t^T$ after using (1.7) and (3.1) to eliminate $u_t$ and $n_t^T$. With $N(x, t) \equiv \langle u(x, t)n^T(t)\rangle$, we get

$$N_t = L[N] + NA^T \qquad (x \in V,\, t > 0) \tag{3.6}$$

where use has been made of (3.5). Equation (3.6) is supplemented by the initial condition

$$N(x, 0) = 0 \qquad (x \in \bar{V}) \tag{3.7}$$

and the boundary condition

$$B_x[N] = E(x, t)\langle n(t)n^T(t)\rangle \equiv E(x, t)\Sigma(t), \qquad (x \in S) \tag{3.8}$$

which follow from (1.8) and (1.9), respectively. Equations (3.6)–(3.8) completely determine $N(x, t)$.

Having obtained $N(x, t)$, we then use (2.3)–(2.6) to determine $U(x, y, t)$, where $F(x, y, t) = E(x, t)N(y, t)$ which does not vanish for $y \in V$, in contrast to (2.4) for temporally uncorrelated data.

### 3.2 *The autocorrelation matrix*

For boundary data which are shaped filtered shot noise, the autocorrelation matrix of $u$ still satisfies equations (2.17) and (2.19). But equation (2.18) is replaced by

$$B_{xt}[R(x, t; y, \tau)] = E(x, t)C(t; y, \tau), \qquad (x, y) \in (S \times \bar{V}) \tag{3.9}$$

where $C(t; y, \tau) = \langle n(t)u^T(y, \tau)\rangle$. The right hand side of (3.9) does not vanish for $t > \tau$ since $n(t)$ is no longer uncorrelated.

To determine $C(t; y, \tau)$, we simply multiply (3.1) by $u^T(y, \tau)$ and ensemble average to get

$$C_t = A(t)C \qquad (y \in \bar{V}, t > \tau) \tag{3.10}$$

where use has been made of the fact that, for $t > \tau$,

$$\langle w(t)u^T(y, \tau)\rangle = \int_S \int_0^\tau \langle w(t)n^T(\tau')\rangle E^T(y', \tau')G^T(y, \tau \mid y', \tau')\, d\tau'\, dy' \tag{3.11}$$

vanishes in view of (3.4). The ordinary differential equation (3.10) is supplemented by the initial condition

$$C(\tau; y, \tau) = N(y, \tau) \tag{3.12}$$

which follows from the definition of the two quantities involved. Since $N(y, \tau)$ has already been determined by (3.6)–(3.8), equations (3.10) and (3.12) define $C(t; y, \tau)$ for $t > \tau$ and $y \in \bar{V}$. With this result, equations (2.17), (2.19) and (3.9) now determine $R(x, t; y, \tau)$.

From the point of view of a numerical solution for $R(x, t; y, \tau)$, the procedure outlined above is much more efficient than the autocorrelation method described in section (2.2) since it takes advantage of the fact that $n(t)$ is a shaped filtered shot noise process. In particular, the determination of the cross-correlation $C$ involves only the solution of an IVP in ODE and consumes only a tiny fraction of the computing time required by the corresponding step in the autocorrelation method.

### 3.3 *Steady state meansquare temperature distribution in a rod with an exponentially correlated end flow rate*

We now apply the results of section (3.1) to the one dimensional heat flow problem (1.2)–(1.4) with a zero mean, exponentially correlated $f(t)$,

$$\langle f(t)f(\tau)\rangle = \sigma_0 e^{-\alpha|t-\tau|}, \tag{3.13}$$

where $\alpha > 0$ and $\sigma_0 > 0$ are positive constants. In the process of obtaining the steady state meansquare temperature distribution of the rod, we will bring out some additional features of our method.

For the purpose of obtaining the second order statistics of $u$, we may think of $f(t)$ as the steady state (stationary) output of a dynamical system characterized by the ordinary differential equation

$$f_t + \alpha f = \sqrt{(2\alpha)}w(t) \tag{3.14}$$

where $w(t)$ is a zero mean, ideal white noise process with $\langle w(t)w(\tau)\rangle = \sigma_0\,\delta(t-\tau)$. We can show by way of the representation

$$f(t) = \sqrt{(2\alpha)}\int_{-\infty}^{t} e^{-\alpha(t-z)}w(z)\,\mathrm{d}z \tag{3.15}$$

that the autocorrelation of $f$ is given by (3.13) and that

$$\langle f(t)w(\tau)\rangle = H(t-\tau)\sqrt{(2\alpha)}\sigma_0\,e^{-\alpha(t-\tau)} \tag{3.16}$$

For the present problem $f(t) \equiv n(t)$ and (3.6)–(3.8) become

$$F_t - F_{xx} + \alpha F = 0 \tag{3.17}$$

$$F(x, 0) = 0, \qquad F(0, t) = 0, \qquad F_x(1, t) = \sigma_0. \tag{3.18}$$

The solution of (3.17) and (3.18) is to be used in (2.24) and (2.25) which, together with (2.22) and (2.23), completely specify $U(x, y, t)$.

However, for the purpose of getting the steady state meansquare (stationary) response, we note that the steady state solution $\bar{F}$ of (3.17) and (3.18) is independent of $t$ and can be obtained from the two point BVP

$$\bar{F}_{xx} - \alpha\bar{F} = 0, \qquad \bar{F}(0) = 0, \qquad \bar{F}_x(1) = \sigma_0. \tag{3.19}$$

The solution of this steady state version of (3.17) and (3.18) is

$$\bar{F}(x) = \frac{\sigma_0}{\sqrt{\alpha}}\,\frac{\sinh\sqrt{(\alpha x)}}{\cosh\sqrt{\alpha}}. \tag{3.20}$$

This solution is then used in the steady state version of (2.22)–(2.25),

$$\bar{U}_{xx} + \bar{U}_{yy} = 0 \tag{3.21}$$

with

$$\bar{U}(0, y) = 0, \qquad \bar{U}_x(1, y) = \bar{F}(y) \qquad (0 \le y \le 1) \tag{3.22}$$

$$\bar{U}(x, 0) = 0, \qquad \bar{U}_y(x, 1) = \bar{F}(x) \qquad (0 \le x \le 1) \tag{3.23}$$

to determine the steady state spatial correlation function.

The solution of the above BVP by separation of variables is

$$\bar{U}(x, y) = 2\sigma_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n^2 + \alpha)\lambda_n \cosh\lambda_n}\,[\sinh(\lambda_n x)\sin(\lambda_n y) + \sinh(\lambda_n y)\sin(\lambda_n x)] \tag{3.24}$$

where $\lambda_n = \frac{1}{2}(2n+1)\pi$. With $\sigma_0 = \alpha/2$, we see that this solution reduces to the solution (2.30) for a white noise $f(t)$ as $\alpha \to \infty$. It is also not difficult to verify that the series (3.24) tends to the series representation of $\sigma_0 xy$ (as it should) as $\alpha \to 0$. Thus, *the meansquare temperature distribution is nearly parabolic in $x$ for $\alpha \ll 1$.* For moderate values of $\alpha$, the series (3.24) converges so rapidly that $\bar{U}$ may be approximated by the first term of the series. Again, this single series is less cumbersome than the double Fourier series solution by the conventional Green's function method.

For a more complex response process which is stationary in its steady state, the fact that we may get its steady state spatial correlation by solving two BVP numerically (instead of two IBVP) constitutes an additional improvement in computational efficiency of the spatial correlation method.

## 4. GENERAL BOUNDARY DATA

The step from the results of section 3 to an efficient numerical solution scheme for more general space–time random boundary data is essentially parallel to that for random distributed loading described in[5]. No new element appears in the analysis for prescribed random boundary values. Therefore, we will not report the details here but merely refer readers to[5].

## 5. CONCLUDING REMARKS

While we have demonstrated in sections 2.3 and 3.3 (as well as in[4] and[5]) the simplicity of the spatial correlation method for simple problems, the usefulness of the method still lies in its effectiveness for more complex dynamic processes such as the forced random vibration of flexible lifting rotor blades for which conventional methods are either inapplicable or impractical. For these more complex processes, a numerical solution of the mean-square response by the spatial correlation method requires only a tiny fraction of the computing time needed by other available methods such as the autocorrelation method described in section (2.2) and in[3–5]. More specific data on the reduction of computing time for the case of distributed loading can be found in[6, 7].

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**Абстракт** — В области механики сплошной среды со случайными граничными данными дается формулировка пространственного корреляционного метода для линейных, динамических задач. Весьма важным признаком метода является формулировка нестохастической, смешанной, начально-краевой задачи для предложенной в виде матрицы пространственной функции корреляции вектора параметра состояния. В этих случаях, когда для стохастической задачи не получается функция Грина в виде известных функций, несколько стократ более полезным методом, по сравнению с другими, доступными методами, оказывается численное решение среднеквадратического поведения и другого поведения второго порядка для совокупности статистических результатов, на основе корреляционного метода. Обращается, также внимание на дополнительные усовершенствования коэффициента эффективности вычислительной техники в процессе расчета стационарного поведения, на основе предложенного метода.